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In a photonic realization of qubits the implementation of quantum logic is rather difficult due the extremely weak interaction on the few photon level. On the other hand, in these systems interference is available to process the quantum states. We formalize the use of interference by the definition of a simple class of operations which include linear optical elements, auxiliary states and conditional operations.

We investigate an important subclass of these tools, namely linear optical elements and auxiliary modes in the vacuum state. For this tools, we are able to extend a previous quantitative result, a no-go theorem for perfect Bell state analyzer on two qubits in polarization entanglement, by a quantitative statement. We show, that within this subclass it is not possible to discriminate unambiguously four equiprobable Bell states with a probability higher than 50 %.

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I. INTRODUCTION

In the theory of quantum information processing we are working with the abstract notion of qubits. These are two-level quantum systems on which we can perform either individual operations or joint operations, for example two-qubit gates like the CNOT operation. With these gates it is possible to perform an arbitrary operation on a collection of qubits.

In the field quantum communication we are interested in a photonic implementation of qubits since photons are readily transported through free space or optical fibers. However, the implementation of quantum gates is rather difficult unless it becomes viable to map the photonic qubits onto atomic states which can be more easily processed [1]. From an experimental point of view it would be ideal to process the photonic qubits directly without the need to store them. At the level of single photons it appears to be beyond present experimental capabilities to use non-linear effects to perform two-qubit gate operations.

The implementation of qubits by photons opens another way to process the represented quantum information. Since the qubits are now indistinguishable particles, we can use interference by mixing the photonic modes us-

ing linear optical elements¹. In general, we are thereby leaving the simple picture of two-level systems, but if we are interested purely in the extraction of information via measurements on some qubits, rather than operations on qubits, then this is not a drawback at all. In this paper we define a simple class of tools which is designed to capture elements of a simple implementation of general measurement on photonic qubits. It comprises mixing of optical modes via linear optical elements, the use of auxiliary modes prepared in an arbitrary initial state, the use of conditional dynamics which allows to control the evolution of a subsystem conditioned on the measurement result on some other subsystem and, finally, photon-counting measurements.

The investigation of the measurements implementable with these tools is important because of the direct relation to simple physical implementations. Additionally, it is even very important to investigate the power of subclasses of these tools. For example, measurements which can be performed with auxiliary modes prepared in the vacuum state and which do not need to make use of conditional dynamics are the easiest measurements one can think of. It is therefore important to see what basic class of measurements can thus be realized, and how the additional tools, like non-vacuum auxiliary modes or conditional dynamics, extend our capability to perform non-trivial measurements.

An essential measurement in quantum communication is the Bell measurement on a product of two separate qubit systems. This measurement is an essential tool for teleportation [2], dense coding [3], quantum repeaters [4] and fault tolerant quantum computing [5]. We have shown earlier [6] that it is not possible to perform a perfect Bell measurement with these tools on two qubits represented by the polarization state of a photon. However, this leaves room for an apparatus which unambiguously discriminates between the four Bell states with probability less than one.² In the present paper, we investigate the restricted problem where we make use only of

¹In this paper we will refer to linear optical elements meaning *passive* linear optical elements which mix linearly the mode creation operators (see Eq. (1)). In contrast *active* linear optical elements mix linearly creation and annihilation operators.

²Actually, as we finish this manuscript, a work by Knill and coworkers [7] presents a way to approach unit probability of success asymptotically with a growing number of entangled auxiliary photons, making full use of all presented tools.

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vacuum state auxiliary modes and exclude conditional dynamics. The aim is to provide an upper bound on the success probability of a Bell measurement, using the restricted set of tools. Our result shows that it is possible for equiprobable Bell states to obtain an unambiguous result in half of the cases only, showing that the early proposed implementations [8] are indeed optimal within this class.

II. DESCRIPTION OF VIABLE MEASUREMENTS

Before we continue we shall describe our tools more precisely. We restrict our measurement apparatus to linear elements. This means that the creation operators of the output modes ($\{c_i^\dagger\}$) are a linear combination of the input modes ($\{a_i^\dagger\}$),

$$c_i^\dagger = \sum_{j=1}^n U_{ij}^\dagger a_j^\dagger \quad (1)$$

where U is a unitary matrix. Reck *et al.* [9] have shown that in fact any unitary mapping can be realized using only beamsplitters and phase shifters.

The number of modes is not necessarily limited by the number of modes occupied by the input states: we can couple to auxiliary modes using beam-splitters so that the initial state of input and apparatus is described by the direct product of the Hilbert space of the input states and the initial state of the auxiliary modes. The auxiliary modes can be prepared in an arbitrary state with any number of excitations.

All modes are mapped into output modes, where we place detectors. Detectors with photon number resolution are difficult to realize. Nevertheless, since here we want to concentrate on the real power of linear elements we will work with ideal detectors, so that each measurement outcome corresponds to projection onto photon number states.

Here it is important to emphasize that linear optical elements can provide any arbitrary unitary mapping only over creation operators, not over a general input state. Only in the case of one-photon states a unitary map on the creation operators translates into the same map on the one-photon states. In fact, making use of this result and of the Neumark extension [10,11], we can realize any generalized measurement over one-photon states using the tools described so far.

The last tool introduced here is the ability to perform conditional measurements. With that we mean that we monitor one selected mode while keeping the other modes in a waiting loop. Then we can perform some linear operation on the remaining modes depending on the outcome of the measurement with all the tools described above.

The general strategy is shown schematically in figure 1 and makes use of electronically switched passive linear

optical elements, auxiliary modes with or without excitations and ideal photo-detectors.

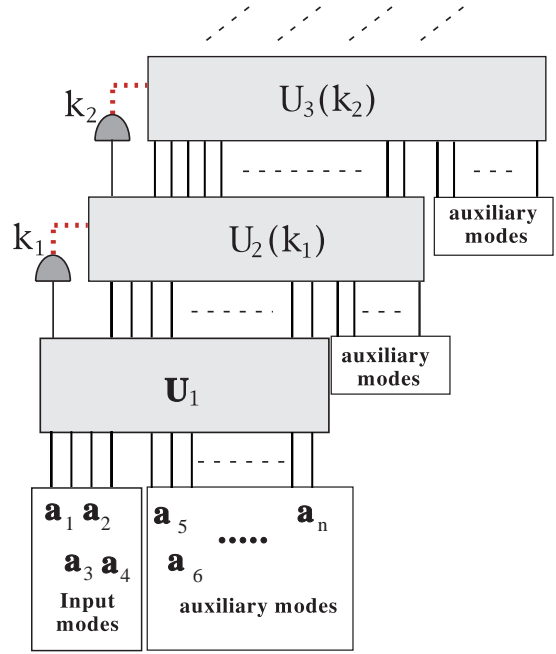


FIG. 1. In the general scheme the modes of the input state are mixed with the auxiliary modes, which are not necessarily in the vacuum state. Then one selected mode is measured and, depending on the measurement outcome, the other output modes are mixed linearly with new auxiliary modes. This procedure can be repeated indefinitely.

III. BELL-STATE ANALYZER EFFICIENCY

A general measurement is described by a positive operator valued measure (POVM) [11] given by a collection of positive operators F_k with $\sum_k F_k = \mathbb{1}$. Each operator F_k corresponds to one classically distinguishable measurement outcome (e.g. a given combination of “clicks” in the output detectors). The probability p_k for the outcome k to occur while the input is being described by density matrix ρ , is given by $p_k = \text{Tr}(\rho F_k)$.

A perfect Bell-state analyzer is an apparatus which performs a von Neumann projection measurement on a maximally entangled basis. That is, it is an apparatus for which every measurement outcome is described by a POVM element proportional to a projector on one of the four orthogonal Bell-states. It has been proven already [6] that with the tools described in the previous section it is not possible to construct such an apparatus.

Here we will relax a bit the constraints of the perfect Bell-state analyzer by allowing it to fail with some probability. Our non-perfect Bell-state analyzer is then defined as a measurement apparatus for which *some* POVM elements $\{F_l\}$ are proportional to a projector onto one of the four orthogonal Bell-states. This means that the

events corresponding to each of these POVM elements can only be triggered by only one of the input Bell-states ($p_l = \text{Tr}(\rho_{\Psi_l} F_l) \neq 0$ for only one of the four Bell state inputs ρ_{Ψ_i} , $i = 1, \dots, 4$). This allows us to rephrase the problem as one of unambiguous state discrimination between the four Bell-states.

In this paper we determine the maximum probability of success for the Bell-state analyzer using a restricted, more basic, set of tools (see fig. 2). Our apparatus consists of a fix array of optical elements (no conditional measurements allowed) and the auxiliary modes are initially in the vacuum state (no extra photons allowed). In the remaining of the paper we will proof that for equiprobable input Bell-states the maximal probability of success of this Bell state analyzer is $P^s = \frac{1}{2}$.³

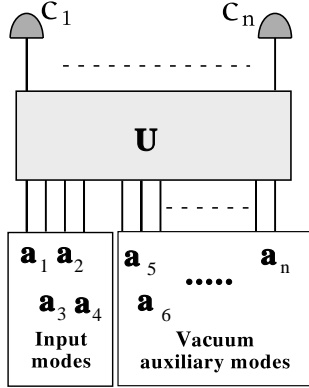


FIG. 2. In the restricted scheme, the modes of the two-photon input states are linearly mixed with the auxiliary vacuum modes. Ideal detectors are then placed in the output modes.

We can summarize the proof as follows: We introduce the formalism which we will use along the proof. We prove that all two-photon detection events are a failure (they do not contribute to the success probability). Then we put an upper bound on the probability that an output mode c_i is involved in a successful discrimination event. In order to fix this upper bound we first show that one mode can participate in the discrimination of at most two Bell states. In the end we obtain the upper bound on the total probability of success by adding the contributions of all modes.

We start by writing up the Bell-states encoded in the polarization degree of freedom of two photons with different momentum. In terms of the creation operators of the input modes they take the form

³This result can be easily generalized to the case where all Bell-states have a non-zero a priori probability but they are not equiprobable (i.e. $p_1 \geq \dots \geq p_4 \neq 0$) giving an optimum success probability of $P^s = p_1 + p_2$.

$$|\Psi^1\rangle = \frac{1}{\sqrt{2}} (a_1^\dagger a_3^\dagger + a_2^\dagger a_4^\dagger) |\mathbf{0}\rangle \quad (2)$$

$$|\Psi^2\rangle = \frac{1}{\sqrt{2}} (a_1^\dagger a_3^\dagger - a_2^\dagger a_4^\dagger) |\mathbf{0}\rangle \quad (3)$$

$$|\Psi^3\rangle = \frac{1}{\sqrt{2}} (a_1^\dagger a_4^\dagger + a_2^\dagger a_3^\dagger) |\mathbf{0}\rangle \quad (4)$$

$$|\Psi^4\rangle = \frac{1}{\sqrt{2}} (a_1^\dagger a_4^\dagger - a_2^\dagger a_3^\dagger) |\mathbf{0}\rangle \quad (5)$$

where $|\mathbf{0}\rangle$ is the vacuum state, a_1^\dagger and a_2^\dagger (a_3^\dagger and a_4^\dagger) correspond the two polarization modes of the first (second) photon.

Any two-photon state which enters the apparatus can be defined with a bilinear form,

$$|\Psi\rangle^{in} = \sum_{i,j=1}^n N_{ij} a_i^\dagger a_j^\dagger = \mathbf{a} \cdot \mathbf{N} \cdot \mathbf{a} |\mathbf{0}\rangle \quad (6)$$

where \mathbf{N} is a $n \times n$ symmetric matrix and $\mathbf{a} = (a_1^\dagger, \dots, a_n^\dagger)$ with the elements a_i^\dagger being the bosonic creation operators for the input modes.

Since the input and output modes are related through a linear transformation (Eq. 1) we can write a similar expression for the state in terms of the output modes $\{c_i^\dagger\}$,

$$|\Psi\rangle^{in} = \mathbf{a} \cdot \mathbf{N} \cdot \mathbf{a} |\mathbf{0}\rangle = \mathbf{c} \cdot U^T \mathbf{N} U \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{M} \cdot \mathbf{c} \quad (7)$$

where $\mathbf{c} = (c_1^\dagger, \dots, c_n^\dagger)$ and

$$\mathbf{M} = U^T \mathbf{N} U. \quad (8)$$

Using the previous relation we can write the output bilinear form M corresponding to each of the possible input Bell-States ($\{|\Psi^\mu\rangle\}$),

$$\mathbf{M}^\mu = U^T \mathbf{N}^\mu U \quad \text{where } \mu = 1, \dots, 4 \quad (9)$$

and from Eqs. (2)-(5) the input bilinear forms are,

$$\mathbf{N}^\mu = \frac{1}{2\sqrt{2}} \left(\begin{array}{c|ccc} \mathbf{W}^\mu & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & 0 \end{array} \right). \quad (10)$$

where

$$\mathbf{W}^\mu = \begin{pmatrix} 0 & 0 & \delta_{\mu 1} + \delta_{\mu 2} & \delta_{\mu 3} + \delta_{\mu 4} \\ 0 & 0 & \delta_{\mu 3} - \delta_{\mu 4} & \delta_{\mu 1} - \delta_{\mu 2} \\ \delta_{\mu 1} + \delta_{\mu 2} & \delta_{\mu 3} - \delta_{\mu 4} & 0 & 0 \\ \delta_{\mu 3} + \delta_{\mu 4} & \delta_{\mu 1} - \delta_{\mu 2} & 0 & 0 \end{pmatrix}. \quad (11)$$

We can then rewrite Eq. (9) as,

$$\mathbf{M}^\mu = \frac{1}{2\sqrt{2}} U_{tr}^T \mathbf{W}^\mu U_{tr} \quad (12)$$

where U_{tr} is the truncated version of U obtained by taking only its first four rows (i.e. $\dim(U_{tr}) = 4 \times N$). The matrices \mathbf{W}^μ , which are the first 4×4 diagonal block of \mathbf{N}^μ , are unitary. This property is characteristic of all the maximally entangled states.

From the output bilinear forms \mathbf{M}^μ we can read out the contributions of the different input Bell states to particular detection events.

We start by studying the two-photon detection events. The probability of having a two-photon detection at mode c_i given the input state $|\Psi\rangle^\mu$ is,

$$P_i^\mu[2] = \langle 0 | c_i^2 M_{ii}^{\mu*} M_{ii}^\mu c_i^\dagger | 0 \rangle = \frac{1}{4} |\alpha_i \cdot W^\mu \cdot \alpha_i|^2 \quad (13)$$

where α_i is the i^{th} column-vector of U_{tr} , i.e. $\alpha_i = \{U_{1i}, \dots, U_{4i}\}$. That is, the vector α_i gives the linear relation between the output-mode c_i^\dagger and the modes where we feed the input states ($a_l^\dagger, l \leq 4$). Since the input modes are coupled to the auxiliary modes, the vectors α_i are in general not orthonormal.

In order to identify unambiguously one Bell-State through a two-photon detection in mode c_i , the probability of this event has to vanish for three of the Bell-States. Using Eq. (13) to impose this condition one can obtain the following solutions for α_i ,

$$\alpha_i = (a, b, 0, 0) \text{ and } \alpha_i = (0, 0, a, b) \quad (14)$$

But it is easy to check that for both solutions the two-photon detection probability vanishes for all Bell-States, i.e. $P[2]_i^\mu = 0$ for $\mu = 1, \dots, 4$. So, a two-photon detection at given output mode c_i cannot identify unambiguously one Bell state. After a two photon detection the remaining state is for all the Bell states the vacuum state, therefore an event of this type counts as a failure of the Bell-State analyzer.

Now we can concentrate on the single-photon detection event. From the result in the previous paragraph we know that these are the only events which can lead to a successful discrimination. After a single photon detection in mode c_i the remaining photon is in the following conditional state,

$$|\Phi_i^\mu\rangle = 2 \sum_{j \neq i}^N M_{ij}^\mu c_j^\dagger |0\rangle = 2(\mathbf{m}_i^\mu \cdot \mathbf{c} - M_{ii}^\mu c_i^\dagger) |0\rangle. \quad (15)$$

Again, the index μ stands for the different input Bell-States, and \mathbf{m}_i^μ is the i th column vector of \mathbf{M}^μ . Using Eq. (12) we can write,

$$\mathbf{m}_i^\mu = \frac{1}{2\sqrt{2}} U_{tr}^T \mathbf{W}^\mu \alpha_i = \frac{1}{2\sqrt{2}} U_{tr}^T \mathbf{s}_i^\mu \quad (16)$$

where $\mathbf{s}_i^\mu = \mathbf{W}^\mu \alpha_i$.

It is straightforward to check that the vectors $\{\mathbf{s}_1^1, \dots, \mathbf{s}_4^4\}$, corresponding to the four input Bell states, are linearly dependent. This is formally expressed as

$$\det(\mathbf{s}_1^1, \dots, \mathbf{s}_4^4) = 0 \implies$$

$$\sum_{\mu=1}^4 b_\mu \mathbf{s}_i^\mu = 0 \quad \text{with at least one } b_\mu \neq 0. \quad (17)$$

Since they also have the same norm,

$$|\mathbf{s}_i^\mu|^2 = \alpha_i^* \mathbf{W}^{\mu\dagger} \mathbf{W}^\mu \alpha_i = |\alpha_i|^2 \text{ for } \mu = 1, \dots, 4 \quad (18)$$

at least two coefficients b_μ in Eq. (17) must be non-zero.

From Eq. (17) and the linearity of Eq. (15) and (16) it follows that the conditional states after a one-photon detection are linearly dependent as well,

$$\sum_{\mu=1}^4 b_\mu |\Phi_i^\mu\rangle = 0 \quad \text{with at least two } b_\mu \neq 0. \quad (19)$$

with the same coefficients as the \mathbf{s}_i^μ dependence. The overlaps of the conditional states are

$$\langle \Phi_i^\eta | \Phi_i^\mu \rangle = 4(\mathbf{m}_i^{\eta*} \cdot \mathbf{m}_i^\mu - M_{ii}^{\eta*} M_{ii}^\mu) \quad (20)$$

or, by making use of Eq. (16),

$$\langle \Phi_i^\eta | \Phi_i^\mu \rangle = \frac{1}{2} (\mathbf{s}_i^{\eta*} \cdot \mathbf{s}_i^\mu - (\alpha_i \cdot \mathbf{s}_i^\eta)^* (\alpha_i \cdot \mathbf{s}_i^\mu)). \quad (21)$$

From the previous equation we can calculate the norm of the conditional states $\{|\Phi_i^\mu\rangle\}$, i.e. the probability of a one-photon detection at mode c_i for each input Bell-state,

$$\begin{aligned} P[1]_i^\mu &= \langle \Phi_i^\mu | \Phi_i^\mu \rangle = \frac{1}{2} (|\mathbf{s}_i^\mu|^2 - |\alpha_i \cdot \mathbf{s}_i^\mu|^2) = \\ &= \frac{1}{2} (|\alpha_i|^2 - |\alpha_i \cdot \mathbf{s}_i^\mu|^2) \end{aligned} \quad (22)$$

It is a well known fact [12] that one cannot discriminate (not even with a small probability) states from a set of linearly dependent vectors. But one should be careful when interpreting this statement. What is true is that a state can be discriminated from a set of vectors if and only if it is linearly independent of this set of vectors, i.e. it can not be written as a linear combination of the vectors in this set. But, according to the definition in Eq.(19) it might well be the case that a vector $|\Phi_i^\mu\rangle$ from a linear dependent set is linearly independent from the other vectors of the set. This would mean that the corresponding factor b^μ is zero. A state with this property can then be discriminated from the rest with a non-zero probability. The fact that the four conditional states in Eq.(19) are linearly dependent with at least two non-zero coefficients b^μ means that the minimum set of linear dependent states contains two states. Consequently the maximum number of states which may be unambiguously discriminated, i.e. which are linearly independent of the others, is two.

Let us call a and b the two different values of the index μ for which the conditional states are linearly independent. These correspond to the two Bell-states that with

some probability may be unambiguously discriminated after a one photon detection in mode c_i . With this result and by assuming that the probability of unambiguous discrimination is one, we can put an upper bound to the *success probability* p_i^s that the detector in mode c_i is involved in the unambiguous discrimination of a Bell-State,

$$p_i^s \leq \frac{1}{4}(P[1]_i^a + P[1]_i^b) \leq \frac{1}{4}|\alpha_i|^2 \quad (23)$$

where the factor $\frac{1}{4}$ is the a priori probability of the corresponding initial Bell-states.

Each successful detection event involves single-photon detections in two different modes, say c_i and c_j . This means that such an event contributes to the corresponding probabilities of success, p_i^s and p_j^s of both modes. This double counting is compensated by a factor $\frac{1}{2}$ when collecting the contributions from all the modes to the total probability of success,

$$P^s \leq \frac{1}{2} \sum_{i=1}^n p_i^s = \frac{1}{8} \sum_{i=1}^n |\alpha_i|^2 = \frac{1}{2}. \quad (24)$$

Here we have made use of,

$$\sum_{i=1}^n |\alpha_i|^2 = \sum_{i=1}^N \sum_{j=1}^4 |\mathbf{U}_{ji}|^2 = \sum_{j=1}^4 1 = 4. \quad (25)$$

This upper bound can be achieved following the scheme proposed in [8], and has been realized experimentally by Mattle *et al.* [13] in the implementation of the quantum dense coding scheme. Using a 50/50 beam-splitter and two polarizing beam-splitters it is possible to unambiguously discriminate the states Ψ^1 and Ψ^2 successfully but Ψ^3 and Ψ^4 give the same measurement outcomes. One therefore obtains a total probability of success of $P^s = \frac{1}{2}$. Since one can transform a Bell-state into any other Bell-state by local unitary transformations it is clear that one can unambiguously discriminate any other two Bell-states. For example, by inserting a $\frac{\lambda}{2}$ plate before one of the input modes of the previous apparatus one would unambiguously discriminate the states Ψ^3 and Ψ^4 . Not all linear-optical Bell-analyzers are restricted to detect two types of Bell states. For example, there exist a fix array of beam splitters and phasers which unambiguously discriminates three Bell-states, but each of them with probability smaller than one so that $P^s \leq \frac{1}{2}$.

IV. CONCLUSIONS

In this paper we have proven that the maximum probability of success of a Bell-state analyzer build with a fix array of linear optical elements is $P^s = \frac{1}{2}$ for equiprobable input Bell-states. Quantitative results for the case where conditional measurements and auxiliary photons can be used have not been given here, but a recent work by Knill, Laflamme and Milburn [7] shows that with these

tools it is in principle possible to perform Bell measurements, among other crucial quantum measurements, with a probability of failure arbitrarily small. The probability of failure of their schemes decreases as $1/(n+1)$ with the number n of highly entangled photons used in the auxiliary modes. This makes their scheme unpracticable with the current technology.

Recently there have been some proposals to realize complete Bell measurements but all of them require non-linear optical elements [15,14] and are still far from being implementable with the current technology.

With this paper we would like to stimulate the study of the capabilities and limitations of quantum information processing with linear elements when qubits are encoded in indistinguishable particles. Only very recently some work has been done in this direction. Bouwmeester [16] has shown a way to implement a scheme which rejects single bit-flip errors using linear-elements and an initial auxiliary GHZ state. Carollo *et al.* [17] have proved the impossibility to realize a complete measurement in the “non-local without entanglement” basis [18].

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